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LETTER TO THE EDITOR

# Are all quantum measurements reducible to local position measurements?

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**Abstract.** A typical quantum measurement process involves two steps. Step one is to separate the initial wavefunction into its spatial components which are then guided towards detectors at different locations. The second step involves the direct detection of the particle by a detector. Such a measurement process presupposes the existence of an interaction which can produce step one. This letter gives a general mathematical formulation of the first step and establishes the existence of a unitary evolution group which produces the separation. The results obtained support the view that quantum measurements are reducible to local position measurements.

It is folklore that all quantum measurements are reducible to position measurements; indeed it is hard to imagine otherwise. A typical example is the Stern–Gerlach setup for a spin measurement. It is from the direct detection of the particle in a particular location that one claims to have measured the spin; one does not measure the spin directly [1, chapter 4]. Another example is measurement of the momentum of a charged particle by its deflection in a magnetic field. Measurements of this kind can be considered to consist of two steps [2]. Step one is to separate the initial wavefunction into its spectral components which are then guided towards detectors at various locations. This step is accomplished by a unitary evolution generated by an appropriate interaction without any wavepacket reduction. The second step involves the direct detection of the particle by one of the detectors. A detector is a device which can ascertain the arrival or non-arrival of a particle at a location where the detector is situated. A detector is therefore a position measuring device. The obvious conclusion which can be drawn from this analysis is that the measurement is executed by a position measurement. In this letter we aim to give a general and rigorous formulation of such a measurement process.

This paragraph summarizes our notation. We shall denote by $E_x$ and $E_p$ the position and momentum spectral measures of the Hilbert space $L^2(\mathbb{R}^n)$. These are defined for any Borel set $\Delta$ of $\mathbb{R}^n$ by

$$E_x(\Delta)\varphi = \chi_\Delta\varphi \quad \forall\varphi \in L^2(\mathbb{R}^n) \qquad E_p(\Delta) = \mathcal{F}^{-1}E_x(\Delta)\mathcal{F}$$

where $\chi_\Delta$ is the characteristic function of the set $\Delta$ and $\mathcal{F}$ is the Fourier transform operator on $L^2(\mathbb{R}^n)$ [3]. By an evolution group we shall mean a strongly continuous group homomorphism from $\mathbb{R}$ into the group of all unitary operators on $L^2(\mathbb{R}^n)$. We denote by $U^0$ the free-particle evolution group defined by

$$U_t^0 = \exp(-\mathrm{i}tH_0/\hbar)$$

where $H_0$ is the self-adjoint operator representing the free-particle Hamiltonian, i.e.

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2.$$

The range of an operator $P$ will be denoted by ran$P$. In particular if $P$ is a projection operator then ran$P$ is the subspace onto which $P$ projects.

We now discuss the spatial separation of the wavefunction. Consider a quantum particle of mass $m$ in three-dimensional motion described by the Hilbert space $L^2(\mathbb{R}^3)$. To begin let us consider a proposition of the particle represented by a projection operator $P$; such a proposition takes the values 0 or 1 and is measurable by a yes–no experiment. Our object is to obtain the expectation value $\langle\Phi \mid P\Phi\rangle$ of the observable $P$ of a particle in the state $\Phi$ by local position measurements. Let $M$ be the subspace associated with $P$ and let $M^\perp$ denote the orthogonal complement of $M$. We then have the decomposition

$$\Phi = \phi + \psi$$

for some $\phi \in M$ and some $\psi \in M^\perp$, giving

$$\langle\Phi \mid P\Phi\rangle = \langle\phi \mid \phi\rangle.$$

Let us define precisely what is meant by step one for the separation of the wavefunction. The intuitive idea is to introduce a unitary evolution generated by some interaction Hamiltonian $H$ which makes $\phi$ and $\psi$ evolve into states in two spatially disjoint regions $\Delta$ and $\Lambda$. Let $U$ be such an evolution group and for each $\varphi \in L^2(\mathbb{R}^3)$ let $\varphi_t$ denote $U_t\varphi$. Then we have

$$\Phi_t = \phi_t + \psi_t.$$

Generally the idea of separation has to be an asymptotic one since a Schrödinger-type of evolution with a positive Hamiltonian generally leads to a spreading of the wavepacket [4]. We shall call $\phi_t$ and $\psi_t$ asymptotically separating [5, 6] if there are disjoint regions $\Delta$ and $\Lambda$ in the configuration space $\mathbb{R}^3$ for which

$$\lim_{t\to\infty} \|E_x(t\Delta)\phi_t\| = \|\phi\| \qquad \lim_{t\to\infty} \|E_x(t\Lambda)\psi_t\| = \|\psi\|. \tag{1}$$

Physically this means that at all sufficiently large times $t$ the components $\phi_t$ and $\psi_t$ will separate and concentrate in the respective regions $t\Delta = \{\, tx \mid x \in \Delta \,\}$ and $t\Lambda = \{\, tx \mid x \in \Lambda \,\}$, with vanishing overlap.

This is similar to the known asymptotic separation, under the free evolution group, of two wavefunctions having disjoint momentum ranges—a result which we state more precisely in the following way (see [5–7]).

*Theorem 1.* For any Borel set $\Delta$ of $\mathbb{R}^n$ and any $\phi \in L^2(\mathbb{R}^n)$

$$\lim_{t\to\infty} \|E_x(t\Delta)U_t^0\phi\| = \|E_p(m\Delta)\phi\|.$$

Let us call the unitary evolution given by $U$ a spatial separation of the state $\Phi$ according to the spectrum of the observable $P$. We now regard this as the definition of the intuitive idea of wavefunction separation involved in step one of the measurement of $P$.

The question now is whether such a step one is always realizable, i.e. whether there exists an evolution group $U$ which satisfies requirements (1). The answer is yes, as we see from the following theorem.

*Theorem 2.* Let $\{\phi_1, \ldots, \phi_k\}$ be a finite orthonormal set in $L^2(\mathbb{R}^n)$ and let $\{\Delta_1, \ldots, \Delta_k\}$ be a set of pairwise disjoint Borel sets, each having non-zero Lebesgue measure. Then there is an evolution group $U$ on $L^2(\mathbb{R}^n)$ with the following properties.

(i) $\lim_{t\to\infty} \|E_x(t\Delta_r)U_t\phi_r\| = 1 \; \forall r \in \{1, \ldots, k\}$.

(ii) the wave operator $\Omega_+$ defined by $\Omega_+ = \text{s-}\lim_{t\to\infty} U_t^* U_t^0$ exists and is unitary.

(iii) Every element of $L^2(\mathbb{R}^n)$ is a scattering state of $U$.

*Proof.* There is no loss of generality in assuming that the sets $\Delta_1, \ldots, \Delta_k$ cover $\mathbb{R}^n$ (for if not, take $\Delta_{k+1} = \mathbb{R}^n - \cup_{r=1}^k \Delta_r$ and add a suitable vector $\phi_{k+1}$).

Extend $\{\phi_1, \ldots, \phi_k\}$ to an orthonormal basis $\{\phi_1, \ldots, \phi_K\}$ of the subspace spanned by the set $\{E_p(m\Delta_r)\phi_s \mid 1 \le r, s \le k\}$. For each $r$ let $\{\phi_{jk+K+r} \mid j = 0, 1, 2, \ldots\}$ be an orthonormal basis for the subspace

$$\{E_p(m\Delta_r)\phi_s \mid 1 \le s \le k\}^\perp \cap \operatorname{ran} E_p(m\Delta_r).$$

In this way we extend the original set of $k$ vectors to an orthonormal basis $\{\phi_i \mid i \in \mathbb{N}\}$ of $L^2(\mathbb{R}^n)$ with the property that there exists $K \ge k$ with

$$E_p(m\Delta_r)\phi_{K+r} = \phi_{K+r} \qquad 1 \le r \le k \tag{2}$$

and

$$E_p(m\Delta_r)\phi_i \in \{0, \phi_i\} \qquad i \ge K. \tag{3}$$

We now map the original set of $k$ vectors to a new set whose elements correspond to disjoint momentum ranges and therefore separate asymptotically under the free evolution.

Let $F$ be the bounded linear operator defined by

$$\begin{aligned}
F\phi_r &= \phi_{K+r} && 1 \le r \le k \\
F\phi_{K+r} &= \phi_r && 1 \le r \le k \\
F\phi_r &= \phi_r && k < r \le K \\
F\phi_r &= 0 && K + k < r
\end{aligned}$$

and let $G$ be the projection onto the subspace $\{\phi_i \mid 1 \le i \le K + k\}^\perp$. Define $W$ to be the operator $F + G$. Then $W$ is unitary ([3, p 215]) and $W^2 = I$, so $W$ is self-adjoint. We may now define an evolution group $U$ by

$$U_t = WU_t^0W \quad \forall t \in \mathbb{R}.$$

Let $r \in \{1, \ldots, k\}$ then

$$\begin{aligned} U_t\phi_r &= WU_t^0\phi_{K+r} \\ &= W\sum_{i=1}^{\infty}\langle\phi_i \mid U_t^0\phi_{K+r}\rangle\phi_i \\ &= \sum_{i=1}^{K+k}\langle\phi_i \mid U_t^0\phi_{K+r}\rangle F\phi_i + \sum_{i=K+k+1}^{\infty}\langle\phi_i \mid U_t^0\phi_{K+r}\rangle\phi_i. \end{aligned}$$

Observe that if $i > K$ and $E_p(m\Delta_r)^\perp\phi_i \neq 0$ then by (2), (3) and the fact that $E_p(m\Delta_r)^\perp$ commutes with $U_t^0$, we have

$$\begin{aligned} \langle\phi_i \mid U_t^0\phi_{K+r}\rangle &= \langle E_p(m\Delta_r)^\perp\phi_i \mid U_t^0\phi_{K+r}\rangle \\ &= \langle\phi_i \mid U_t^0 E_p(m\Delta_r)^\perp\phi_{K+r}\rangle \\ &= 0. \end{aligned}$$

Hence

$$E_p(m\Delta_r)^\perp U_t\phi_r = \sum_{i=1}^{K+k}\langle\phi_i \mid U_t^0\phi_{K+r}\rangle E_p(m\Delta_r)^\perp F\phi_i$$

so

$$\lim_{t\to\infty}\|E_p(m\Delta_r)^\perp U_t\phi_r\| \leq \sum_{i=1}^{K+k}\lim_{t\to\infty}|\langle\phi_i \mid U_t^0\phi_{K+r}\rangle| = 0$$

since every element of $L^2(\mathbb{R}^n)$ is a scattering state of $U_t^0$ [8, p 125]. It follows that

$$\lim_{t\to\infty}\|E_p(m\Delta_r)U_t\phi_r\| = 1 \quad \forall r \in \{1, \ldots, k\}. \tag{4}$$

We shall now prove parts (ii) and (iii) of the theorem and then use them together with (4) to obtain part (i).

Let $H = WH_0W$ so that

$$e^{-H} - e^{-H_0} = We^{-H_0}W - e^{-H_0}.$$

Since $W\phi = \phi$ for any $\phi \in \operatorname{ran}G$ a simple calculation gives

$$(e^{-H} - e^{-H_0})G = (F - G^\perp)e^{-H_0}G$$

and this implies

$$e^{-H} - e^{-H_0} = (e^{-H} - e^{-H_0})G^\perp + (F - G^\perp)e^{-H_0}G.$$

Since $F$ and $G^\perp$ are finite-rank operators it follows that $e^{-H} - e^{-H_0}$ is a finite-rank operator [9, p 65]; it therefore belongs to the trace class [10, p 209], and this implies that the wave operator $\Omega_+$ exists and is complete [11, corollary 4, p 31], i.e. the range of $\Omega_+$ is equal to the set of all scattering states of $H$ and also to the absolutely continuous subspace of $H$. Since $H$ and $H_0$ are unitarily equivalent, and the spectrum of $H$ is absolutely continuous [12, p 593], we have $\mathrm{ran}\Omega_+ = L^2(\mathbb{R}^n)$. It also follows that $\Omega_+$ is unitary [8, proposition 5.12].

Now for each $r \in \{1, \ldots, k\}$ we have

$$\begin{aligned} 1 &= \lim_{t\to\infty} \|E_p(m\Delta_r)U_t\phi_r\| \\ &= \lim_{t\to\infty} \|E_p(m\Delta_r)U_t^{0*}U_t\phi_r\| \\ &= \|E_p(m\Delta_r)\Omega_+^*\phi_r\| \\ &= \lim_{t\to\infty} \|E_x(t\Delta_r)U_t^0\Omega_+^*\phi_r\| \end{aligned}$$

by theorem 1, but

$$\begin{aligned} \lim_{t\to\infty} \|E_x(t\Delta_r)U_t\phi_r - E_x(t\Delta_r)U_t^0\Omega_+^*\phi_r\| &\le \lim_{t\to\infty} \|\phi_r - U_t^*U_t^0\Omega_+^*\phi_r\| \\ &= \|\phi_r - \Omega_+\Omega_+^*\phi_r\| \\ &= 0 \end{aligned}$$

so

$$\lim_{t\to\infty} \|E_x(t\Delta_r)U_t\phi_r\| = 1.$$

□

We now discuss local position measurement. The position spectral projection $E_x(\Delta)$ corresponds to the proposition 'the measured value of position is in the region $\Delta$' [13], which is a proposition measurable by a detector covering the region $\Delta$. We shall call $E_x(\Delta)$ a local position observable.

Let us return to the problem of the measurement of the projection $P$ in the state $\phi$. The theorem above ensures the existence of two disjoint spatial regions $\Delta$ and $\Lambda$, and an evolution group $U$ such that the conditions (1) are satisfied. From the expansion

$$\begin{aligned} \langle\Phi_t \mid E_x(t\Delta)\Phi_t\rangle = \langle\phi_t \mid E_x(t\Delta)\phi_t\rangle + \langle\psi_t \mid E_x(t\Delta)\psi_t\rangle + \langle\phi_t \mid E_x(t\Delta)\psi_t\rangle \\ + \langle\psi_t \mid E_x(t\Delta)\phi_t\rangle \end{aligned}$$

we deduce that

$$\lim_{t\to\infty} \langle\Phi_t \mid E_x(t\Delta)\Phi_t\rangle = \lim_{t\to\infty} \langle\phi_t \mid E_x(t\Delta)\phi_t\rangle = \|\phi\|^2.$$

We conclude that the expectation value

$$\langle\Phi \mid P\Phi\rangle = \langle\phi \mid \phi\rangle = \|\phi\|^2$$

of $P$ in the state $\Phi$ can indeed now be obtained with arbitrary accuracy by a measurement of the local position observable $E_x(T\Delta)$ in the evolved state $\Phi_T$ after a sufficiently large time interval $T$. In fact we can break up the region $T\Delta$ into a number of smaller regions$(T\Delta)_j$ and employ smaller detectors to obtain the desired expectation value since

$$\lim_{t\to\infty} \langle\Phi_t \mid E_x(t\Delta)\Phi_t\rangle = \sum_j \lim_{t\to\infty} \langle\Phi_t \mid E_x((t\Delta)_j)\Phi_t\rangle.$$

Since an arbitrary observable can be described as a proposition-valued measure and all measurements are reducible to a set of yes–no experiments [13] the fact that the measurement of propositions can be reduced to local position measurements establishes the principle we set out to seek. We can carry out a quantum measurement like a typical scattering experiment. We first carry out a spatial separation of the state according to the spectrum of the observable; the particle is then detected by an array of detectors at appropriate locations and the expectation values can be computed from the data recorded.

Further work is in progress to develop a more complete analysis of the problem posed by the present letter.

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